Università di Bologna

Calculus @ QFinance

Lesson 2.8

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Exercises

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Exercise 1 Solve the Cauchy problem

$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \sin x & x \in \mathbb{R} \end{cases} \tag{1}$$

then use formula $(\mathrm{H_s})$ to infer the integration formula

$$\int_{-\infty}^{+\infty} e^{-s^2} \cos(as)\,\mathrm{d}s = \sqrt{\pi}\, e^{-a^2/4},\ \forall a \in \mathbb{R}$$

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} \sin(x + 2s\sqrt{t}) \mathrm{d}s$$

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Use $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ so that the integrand is

$$e^{-s^2}\sin(x+2s\sqrt{t}) = e^{-s^2}\left(\sin x\cos(2s\sqrt{t}) + \cos x\sin(2s\sqrt{t})\right)$$

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then observe that $s\mapsto e^{-s^2}\cos x\sin(2s\sqrt{t})$ is odd, thus

$$u(x,t)=\frac{\sin x}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-s^2}\cos(2s\sqrt{t})\mathrm{d}s$$

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We have shown that $y(t) = \displaystyle\int_{-\infty}^{\infty} e^{-s^2} \cos(2s\sqrt{t}) \mathrm{d}s$ solves the initial value problem

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$$\begin{cases} y'(t) = -y(t) \\ y(0) = \sqrt{\pi} \end{cases}$$

therefore $y(t) = \sqrt{\pi} e^{-t}$ and then $u(x,t) = \sin x e^{-t}$

Exercise 2 Solve the Cauchy problem

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Bernoulli's equation: we seek for a solution of the form $y(x) = c(x)e^{A(x)}$ being $c(x)$ a function to be determinated and $A(x) = \int_1^x (-1)dz = 1 - x$

So our candidate solution is $y(x) = c(x)e^{1-x}$ and we evaluate

$$y' + y - xy^3$$

obtaining

$$e^{1-x}c'(x) - e^{3-3x}xc^3(x)$$

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$$c'(x) = e^{2-2x}xc^3(x) \tag{2b}$$

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to associate the proper initial condition at (2b) recall form (2) that $y(1) = 2$ and that $y(1) = c(1)e^0 = c(1)$

$$\begin{cases} c'(x) = e^{2-2x} x c^3(x) \\ c(1) = 2 \end{cases} \qquad \Longrightarrow \int_2^c \frac{1}{z^3} \mathrm{d}z = \int_1^x s e^{2-2s} \mathrm{d}s \qquad (2\mathrm{b}_1)$$

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$$\frac{1}{8} - \frac{1}{2c^2} = \frac{3}{4} - \frac{1}{4} e^{2-2x}(2x+1)$$

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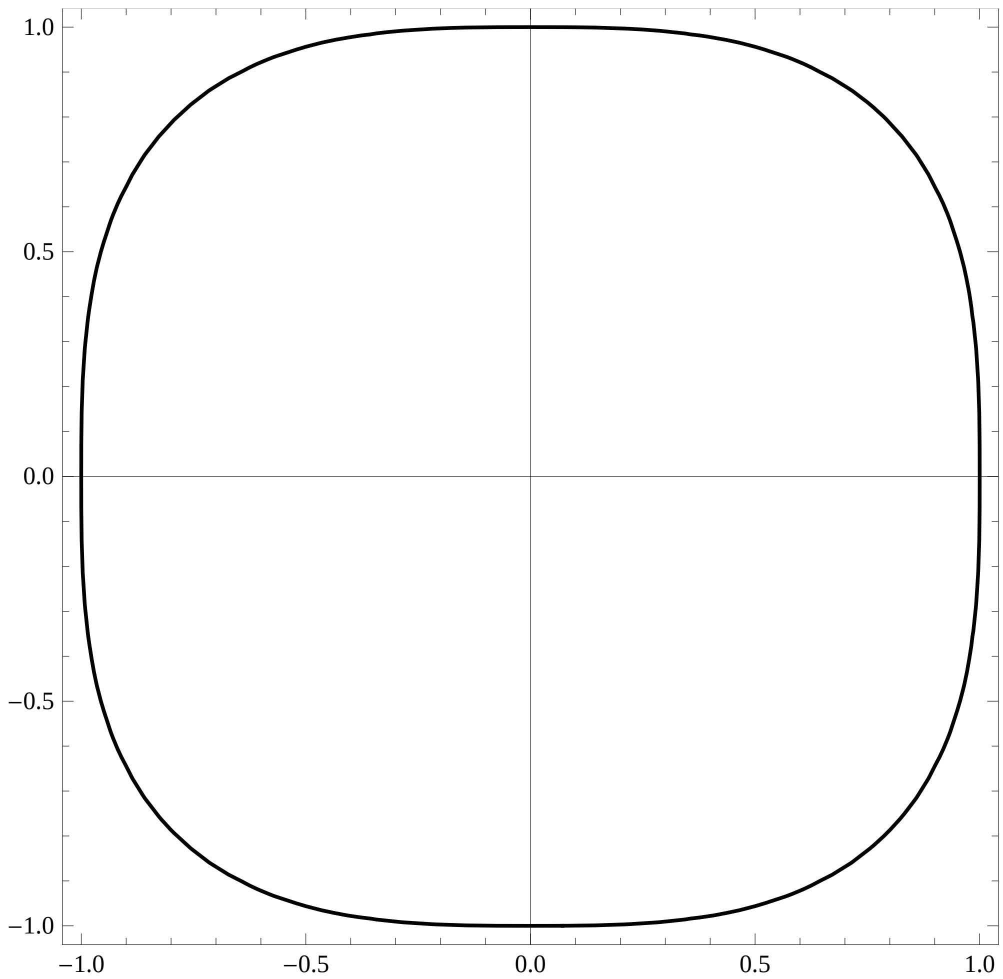
conclusion

$$y(x) = e^{1-x} \sqrt{\frac{4e^{2x-2}}{4x + 2 - 5e^{2x-2}}}$$

Exercise 3 Find the Lebesgue measure of the set

$$A = \{(x, y) \in \mathbb{R}^2 : |x|^3 + |y|^3 \leq 1\}$$


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

$$\ell(A) = 4\int_0^1 \left(\int_0^{(1-x^3)^{1/3}} \mathrm{d}y \right) \mathrm{d}x$$

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$$x^3 = s \implies \mathrm{d}x = \frac{1}{3}s^{-2/3} \implies \ell(A) = \frac{4}{3}\int_0^1 (1-s)^{1/3}s^{-2/3}\mathrm{d}s$$

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$$\ell(A) = \frac{4}{3} B\left(\frac{4}{3}, \frac{1}{3}\right) = \frac{4}{3} \frac{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{3})}$$

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but $\Gamma(\frac{4}{3}) = \frac{1}{3}\Gamma(\frac{1}{3})$, $\Gamma(\frac{5}{3}) = \frac{2}{3}\Gamma(\frac{2}{3})$ thus $\ell(A) = \frac{2}{3}\frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})}$

we are not finished . . .

and so

$$\Gamma(\frac{2}{3}) = \frac{2\pi}{\sqrt{3}} \left(\Gamma(\frac{1}{3}) \right)^{-1}$$

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$$\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

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conclusion

$$\ell(A) = \frac{\Gamma^3(\frac{1}{3})}{\pi\sqrt{3}}$$

Exercise 4 Find the distance

$$d(X,Y) := \inf_{\boldsymbol{x}\in X,\, \boldsymbol{y}\in Y} ||\boldsymbol{x} - \boldsymbol{y}||$$

between $Y = \{(1,1)\}$ and $X = \{(x,y) \in \mathbb{R}^2 : \; x^2 + xy + y^2 = 1\}$

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$$\begin{cases} 2(-1+x) - m(2x+y) = 0 \\ 2(-1+y) - m(x+2y) = 0 \\ x^2 + xy + y^2 - 1 = 0 \end{cases}$$

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$$\begin{cases} 2(-1+x) - m(2x+y) = 0 \\ 2(-1+y) - m(x+2y) = 0 \\ x^2 + xy + y^2 - 1 = 0 \end{cases} \implies \begin{cases} x = \dfrac{2}{2-3m} \\ y = \dfrac{2}{2-3m} \\ \dfrac{12}{(2-3m)^2} - 1 = 0 \end{cases}$$

It follows $2 - 3m = \pm 2\sqrt{3}$ and $x = y = \pm\dfrac{1}{\sqrt{3}}$ the minimum iss assumed for $+$ and its value is $\dfrac{8}{3} - \dfrac{4}{\sqrt{3}}$

Exercise 5

$A := \left\{(x, y) \in R^2 : \; x^2 + 2y^2 \leq 1\right\}$, $f(x, y) = \dfrac{1}{1 + x^2 + 2y^2}$. Evaluate:

$$\mathcal{I} = \iint_A f(x, y) \mathrm{d}x \mathrm{d}y$$

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$$\begin{cases} x = \rho \cos \vartheta \\ y = \dfrac{1}{\sqrt{2}} \rho \sin \vartheta \end{cases}$$

$$
\begin{aligned}
\mathcal{I} &= \frac{1}{\sqrt{2}} \iint_{]0,1]\times[0,2\pi[} \frac{\rho}{1+\rho^2} d\rho d\vartheta \\
&= \frac{2\pi}{\sqrt{2}} \int_0^1 \frac{\rho}{1+\rho^2} d\rho \\
&= \frac{2\pi}{\sqrt{2}} \frac{1}{2} \left[\ln(1+\rho^2)\right]_0^1 \\
&= \frac{\pi \ln 2}{\sqrt{2}}.
\end{aligned}
$$

Exercise 6

Given $A := \left\{ (x,y) \in \mathbb{R}^2 : \, 1 \leq x \leq e, \frac{\ln x}{x} \leq y \leq \frac{1}{x} \right\}$ evaluate:

$$\mathcal{I} = \iint_A e^{xy} \mathrm{d}x \mathrm{d}y$$

Exercise 6

Given $A := \left\{(x, y) \in \mathbb{R}^2 : 1 \le x \le e, \frac{\ln x}{x} \le y \le \frac{1}{x}\right\}$ evaluate:

$$\mathcal{I} = \iint_A e^{xy} \mathrm{d}x \mathrm{d}y$$

$$\begin{aligned} \mathcal{I} &= \int_1^e \left(\int_{\frac{\ln x}{x}}^{\frac{1}{x}} e^{xy} dy\right) dx = \int_1^e \left[\frac{e^{xy}}{x}\right]_{y=\frac{\ln x}{x}}^{y=\frac{1}{x}} dx \\ &= \int_1^e \left(\frac{e}{x} - 1\right) dx = \left[e \ln x - x\right]_1^e = 1. \end{aligned}$$